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THE VANISHING OF THE HOMOLOGY OF CERTAIN GROUPS OF HOMEOMORPHISMS*

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LET G_n be the group of all homeomorphisms of $\mathbb{R}^n$ with compact support (where the *support*, supp h, of a homeomorphism h of $\mathbb{R}^n$ onto itself is defined to be the closure of

$$\{x \in \mathbb{R}^n : h(x) \neq x\}).$$

In this note, we show that the homology of G_n, considered as a discrete group, vanishes in all dimensions >0, i.e.

$$H_r(G_n, \mathbb{Z}) = 0 \qquad r > 0. \tag{1}$$

It is an interesting problem to try to extend this result to other groups of homeomorphisms. In many interesting cases, it seems to be unknown even whether $H_1(G_n, \mathbb{Z})$ (which is isomorphic to the commutator quotient group of G_n) is null [1], so such an extension would appear to be very difficult.

Let B_n denote the open unit ball in $\mathbb{R}^n$ and let $G_n{}^0$ denote the set of elements of G_n whose support lies in B_n. Let $\iota: G_n{}^0 \to G_n$ denote the inclusion map, and let $\iota_*: H_r(G_n{}^0) \to H_r(G_n)$ denote the induced map.

LEMMA. *ι_* is an isomorphism.*

Proof. We recall that if G is any group, then there is a standard chain complex $C(G)$, whose homology is the homology of G. In "inhomogeneous" form, the complex $C(G)$ may be defined as follows. The group $C_r(G)$ is the free abelian group on the set of all r-tuples $(g_1, \ldots, g_r)$ of elements of G. The boundary operator $\partial: C_r(G) \to C_{r-1}(G)$ is defined by

$$\partial(g_1, \ldots, g_r) = (g_1^{-1}g_2, \ldots, g_1^{-1}g_r) + \sum_{(-1)^j = 1}^{r} (g_1, \ldots, \hat{g}_j, \ldots, g_r).$$

If $c = \sum_{j=1}^{r} k_j(g_{j1}, \ldots, g_{jr})$ is an element of the chain group $C_r(G_n)$, we define the *support*, supp c, by

$$\text{supp } c = \bigcup_{j,i} \text{supp } g_{ji}$$

Thus $c \in C_r(G_n{}^0)$ if and only if supp $c \subseteq B_n$.

Now we show that ι_* is surjective. Let $h \in H_r(G_n)$, and let $c \in C_r(G_n)$ be a cycle representing h. Let $\varphi \in G_n$ be such that φ (supp c) $\subseteq B_n$. Let I_φ be the inner automorphism: $I_\varphi(g) =$

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$\varphi g \varphi^{-1}$. Since any inner automorphism induces the identity on homology, $I_{\varphi *}h = h$. But $I_{\varphi *}h$ is represented by the cycle $I_\varphi c$ and clearly supp $I_\varphi c = \varphi$ supp $c \subseteq B_n$. Hence $h = c_* h'$ where $h' \in H_r(G_n{}^0)$ is the homology class represented by $I_\varphi c$.

Next, we show that ι_* is an injection. Let $h \in H_r(G_n{}^0)$ and suppose that $\iota_* h = 0$. Let c be a cycle in $C_r(G_n{}^0)$ representing h. Since $\iota_* h = 0$, there is a cycle c in $C_{r+1}(G_n{}^0)$ such that $\partial c' = c$. Since supp $c \subseteq B_n$, there exists $\varphi \in G_n$ such that φ is the identity in a neighborhood of supp c and $\varphi(\text{supp } c') \subseteq B^n$. Then $\partial(I_\varphi c') = I_\varphi(\partial c') = I_\varphi c = c$. Since $I_\varphi c' \in C_{r+1}(G_n{}^0)$, this proves that $h = 0$.

Proof of (1). It is easily seen that there exists $k \in G_n$ with the following properties:

(a) $k^j(\bar{B}_n) \cap \bar{B}_n = \varphi, \quad j > 0$.

(b) There is a point p in $\mathbb{R}^n$ such that $k^j(\bar{B}_n) \to p$ as $j \to \infty$.

For any $g \in G_n{}^0$ and $i = 0$ or 1, define $\psi_i(g)$ as follows:

$$\begin{aligned} \psi_i(g)(x) &= k^j g k^{-j}(x) && \text{if } x \in k^j(\bar{B}_n) \text{ and } j \geq i \\ &= x && \text{if } x \notin \bigcup_{j \geq i} k^j(\bar{B}_n). \end{aligned}$$

Note that (a) implies that $k^j(\bar{B}_n) \cap k^{j'}(\bar{B}_n) = \varphi$ if $j \neq j'$, so that $\psi_i(g)$ is well defined. From (b) it follows that $\psi_i(g) \in G_n$. Clearly $\psi_i \colon G_n{}^0 \to G_n$ is a homomorphism for $i = 0, 1$. Moreover, ψ_0 and ψ_1 are conjugate, since $\psi_1(g) = k\psi_0(g)k^{-1}$, so

$$\psi_{0*} = \psi_{1*} \colon H_r(G_n{}^0) \to H_r(G_n).$$

Let $\eta \colon G_n{}^0 \times G_n{}^0 \to G_n$ be defined by $\eta(g, h) = g\psi_1(h)$. For $g, h \in G_n{}^0$, supp $g \cap$ supp $\psi_1(h) = \phi$, since supp $g \subseteq B_n$ and supp $\psi_1(h) \subseteq \bigcup_{j \geq 1} k^j(B_n) \cup p$. Hence g commutes with $\psi_1(h)$. It follows that η is a homomorphism. Let $\Delta \colon G_n{}^0 \to G_n{}^0 \times G_n{}^0$ denote the diagonal homomorphism. Clearly

$$\psi_0 = \eta\Delta.$$

Now we prove the following assertion by induction on r: for $1 \leq s \leq r$, $H_s(G_n) = 0$. We begin the induction at $r = 0$, where the assertion is vacuous. For the inductive step we assume that the assertion is true for $r - 1$. By the lemma, it then follows $H_s(G_n{}^0) = 0$ for $1 \leq s \leq r - 1$. By the Künneth formula and this fact,

$$H_r(G_n{}^0 \times G_n{}^0) = H_r(G_n{}^0) \oplus H_r(G_n{}^0)$$

and if $h \in H_r(G_n{}^0)$ then $\Delta_* h = h \oplus h$. Hence

$$\psi_{0*}h = \eta_* \Delta_* h = \iota_* h + \psi_{1*}h = \iota_* h + \psi_{0*}h$$

Hence $\iota_* h = 0$. From Lemma 1, it then follows that $h = 0$.

Thus, we have shown that $H_r(G_n{}^0) = 0$. From Lemma 1, it follows that $H_r(G_n) = 0$, which completes the induction.

REFERENCE

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